Official Reading 3.1, 3.2 of [Co]
Recommended 12.1 of [CH]

Center of mass: 13.1 of [CH], 3.6 of [Co]

Double integrals:

Last time: took $f(x,y)$ and did definite integration twice to get a number

- Subtlety in 2 dimensions: 2 different ways (orders) to integrate:

1st way

- integrate wrt x to get a fcn of y then integrate wrt y to get a #

2nd way

- integrate wrt y first to get a fcn of x, then wrt x.

Miracle: Get same answer → part of Fubini's Theorem

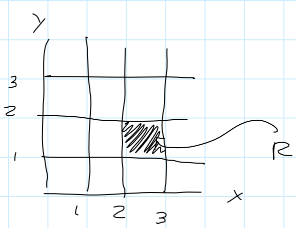
eg. integrate from $x=2$ to $x=3$ & $y=1$ to $y=2$

⇒ integrate over the region:

$$R = [2,3] \times [1,2] = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x \in [2,3] \\ y \in [1,2] \end{array} \right\}$$

just like $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

this is a filled in square



eg. $R = [1,3] \times [1,2]$ → this is a rectangle

Notation

$$\int_{y=1}^{y=2} \int_{x=1}^{x=3} f(x,y)\,dx\,dy = \iint_{R = [1,3]\times[1,2] \subseteq \mathbb{R}^2} f(x,y)\,dx\,dy$$

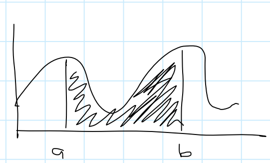
Compare

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx$$

Q/ What does $\iint_R f(x,y)\,dx\,dy$ mean?

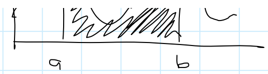
A/ Compare w/ 1 variable:

Consider $\int_{[a,b]} f(x)\,dx$



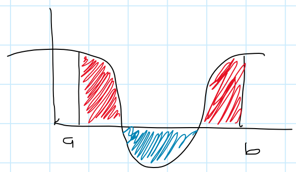
the integral is the

$\int_{[a,b]}$



the integral is the area under the curve

If $f(x) \geq 0$ for $x \in [a,b]$ then it's really the area
But if f goes below the x-axis, then we get signed area



$$\int_{[a,b]} f\,dx = \text{total "signed" area}$$
$$= \text{total area of red regions} - \text{area of blue}$$

Similarly

$\iint_R f(x,y)\,dx\,dy$ is volu

Note when $f(x,y) < 0$, we count the volume as negative.

· ie, we add all volume truly above the xy plane and subtract all volume below

Note if $f(x,y) = 1$, then $\iint_R f\,dx\,dy = \text{area}(R)$

So

we know how to compute $\iint_R f(x,y)\,dx\,dy$ when R is a rectangle with sides parallel to the x and y axes

Then $R = [a,b] \times [c,d]$

and then
$$\iint_R f\,dx\,dy = \int_c^d \int_a^b f\,dx\,dy = \int_a^b \int_c^d f\,dy\,dx$$

(eg if f is continuous)

or piecewise continuous $\longrightarrow$ $f = \begin{cases} \sim & \text{if } \sim \\ \sim & \text{if } \sim \end{cases}$

Upshot For all the fcns we consider in this class, Fubini's Thm is TRUE

Q/ What about a double integral over a more general region?

Recall suppose that we integrate wrt x first. Then you get a fcn

Q/ What about a double integral over a more general region?
<u>Recall</u> suppose that we integrate wrt x first. Then you get a fcn of y and you integrate wrt y

e.g. $$\int_c^d\int_a^b xy\,dx\,dy = \int_c^d \left.\frac{yx^2}{2}\right]_{x=a}^{x=b} dy$$
$$= \int_c^d y\left(\frac{b^2}{2} - \frac{a^2}{2}\right) dy$$
$$= \left[\frac{y^2}{2}\left(\frac{b^2}{2} - \frac{a^2}{2}\right)\right]_{y=c}^{y=d}$$
$$= \frac{(d^2-c^2)(b^2-a^2)}{4}$$

<u>Key</u>: this expr has no x's in it (only y and constants)

Q/ What if we integrated from $x = \frac{y}{2}$ to $x = y$?

A/ Then $\int_{x=\frac{y}{2}}^{y} f(x,y)\,dy$ would still be a fcn of y (w/o x's). And then when we integrate wrt y, we end up with a const.

<u>Here's How</u>

$$\int_{x=\frac{y}{2}}^{x=y} xy\,dx = \left[\frac{yx^2}{2}\right]_{x=\frac{y}{2}}^{x=y}$$
$$= \frac{y(y^2)}{2} - \frac{y(\frac{y}{2})^2}{2}$$
$$= \frac{y^3}{2} - \frac{y^3}{8} = \frac{3y^3}{8}$$

to compute $\iint$ do:

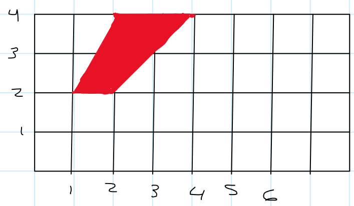
$$\int_{y=c}^{y=d} \frac{3y^3}{8}\,dy = \left[\frac{3y^4}{32}\right]_{y=c}^{y=d}$$
$$= \frac{3(d^4-c^4)}{32}$$

Q/ What is the geometric interpretation?

A/ For each y, we go from $x = \frac{y}{2}$ to $x = y$. Then we add up from $y = c$ to $y = d$

<u>e.g.</u> $c = 2$, $d = 4$

At $y = 2$ we go from $x = 1$ to $x = 2$
At $y = 4$ we go from $x = 2$ to $x = 4$
(at $y = 3$ from $x = \frac{3}{2}$ to $x = 3$)



$$R = \text{red region} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} 2 \le y \le 4 \\ \frac{y}{2} \le x \le y \end{matrix} \right\}$$

is a trapezoid

eg. $$\iint_R xy\,dx\,dy = \frac{3}{32}(d^4 - c^4)$$

$$= \frac{3}{32}(4^4 - 2^4)$$

$$= \frac{3}{2^5}(2^8 - 2^4)$$

$$= \frac{3}{2}(2^3 - 1)$$

$$= \frac{3 \cdot 7}{2} = \frac{27}{2}$$

Q/ Can we do the same integral in the opposite order?

$$\int_2^4 \int_{\frac{y}{2}}^{y} xy\,dx\,dy \overset{?}{=} \int_{\frac{y}{2}}^{y} \int_2^4 xy\,dy\,dx$$

$$= \int_{\frac{y}{2}}^{y} \left. \frac{xy^2}{2} \right]_{y=2}^{y=4} dx$$

$$= \int_{\frac{y}{2}}^{y} 6x\,dx$$

$$= \left. 3x^2 \right]_{\frac{y}{2}}^{y}$$

$$= 3y^2 - \frac{3y^2}{4}$$

→ PROBLEM

this is not a #

For rectangles, we can integrate wrt x or y first.

For region R, it depends on the limits of integration

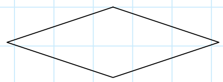
→ If R has 2 vertical sides, you can integrate wrt y first then x

→ If R has 2 horizontal sides, the opposite

Summary

- outer limits of integration must be constants

Q/ What if R is a quadrilateral w/o vertical/horizontal sides?



A/ Break it up into simpler regions and add up the results

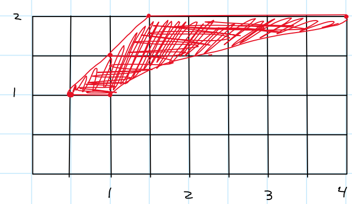
Compare: $\int_a^c f\,dx = \int_a^b f\,dx + \int_b^c f\,dx$

$[a,c] = [a,b] \cup [b,c]$

But first, more examples!

Consider

$$\int_{y=1}^{y=2}\int_{x=\frac{y^2}{2}}^{x=y^2} f\,dx\,dy = \iint_R f\,dx\,dy$$ where R is:



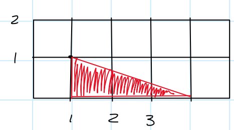
Notice:

non-horizontal sides are curved

Next example: right triangle

hypotenuse: $y = -\frac{x}{2} + \frac{3}{2}$

$x = 3 - 2y$



2 ways to do it

First way: integrate wrt x first

↳ need: two horizontal sides

- bottom horizontal side is the segment $[1,3]$ on x-axis
- top horizontal side is a point $(1,1)$ thought of as a side of length 0
- then if R = the right Δ, then

$$\iint_R f\,dx\,dy = \int_{y=0}^{y=1}\int_{x=1}^{x=3-2y} f\,dx\,dy$$

also think of it as having two vertical sides also think of U as having two vertical sides

→ one side: segment from $(1,0)$ to $(1,1)$

→ another side: point $(3,0)$

get

$$\int_{x=1}^{x=3} \int_{y=0}^{y=-\frac{x}{2}+\frac{3}{2}} f\,dy\,dx = \int \qquad \int$$